Exercise 2.1

Q1 Use the identity \( \tan \theta = \frac{\sin \theta}{\cos \theta} \) to show that \( \frac{\sin \theta}{\tan \theta} - \cos \theta = 0 \).

Q2 Use the identity \( \sin^2 \theta + \cos^2 \theta = 1 \) to show that \( \cos \theta = (1 - \sin \theta)(1 + \sin \theta) \).

Q3 Given that \( x \) is acute, find the exact value of \( \cos x \) if \( \sin x = \frac{1}{2} \).

Q4 Show that \( 4 \sin^2 x - 3 \cos x + 1 = 5 \cos x - 4 \cos^2 x \).

Q5 Given that \( x \) is acute, find the exact value of \( \tan x \) if \( \sin^2 x = \frac{1}{2} \).

Q6 Show that \( \tan x + 1 = \frac{1}{\cos^2 x} \).

Q7 A student is asked to solve the equation \( \sin \theta = \frac{1}{2} \tan \theta \), where \( 0^\circ \leq \theta \leq 90^\circ \).

\[
\begin{align*}
\sin \theta &= \frac{1}{2} \tan \theta \\
\sin \theta &= \frac{1}{2} \times \frac{\sin \theta}{\cos \theta} \\
\sin \theta \cos \theta &= \frac{1}{2} \sin \theta \\
\sin \theta &= \frac{\sin \theta}{2} \\
\text{Find the error they made and explain how this has resulted in an incomplete solution.}
\end{align*}
\]

Q8 Show that \( \tan x + 1 = \frac{1}{\sin x \cos x} \).

Q9 Show that \( 4 + \sin x - 6 \cos^2 x = (2 \sin x - 1)(3 \sin x + 2) \).

Q10 Show that \( \sin^2 x + \cos^2 x = \sin^2 x \sin^2 x \).

Q11 Use the identity \( \sin^2 \theta + \cos^2 \theta = 1 \) to prove Pythagoras' theorem.

3. Trig Graphs

You should be able to draw the graphs of \( \sin x, \cos x \), and \( \tan x \) — including all the important points, like where the graphs cross the axes and their maximum and minimum points.

\( \sin x \)

- The graph of \( y = \sin x \) is periodic — it repeats itself every \( 360^\circ \) (you say it has a period of \( 360^\circ \)).
- So \( \sin x = \sin (x + 360^\circ) = \sin (x + 720^\circ) = \sin (x + 360n^\circ) \), where \( n \) is an integer.
- It bounces between \( y = 1 \) and \( y = 1 \), and it can never have a value outside this range.
- It goes through the origin (as \( x = 0^\circ = 0 \)) and then crosses the \( x \)-axis every \( 180^\circ \).
- \( \sin (-x) = -\sin x \). The graph has rotational symmetry about the origin, so you could rotate it \( 180^\circ \) about \( (0, 0) \) and it would look the same.
- The graph of \( y = \sin x \) looks like this:

\[
\begin{align*}
&m = 0 \\
&y = \sin x
\end{align*}
\]

\( \cos x \)

- The graph of \( y = \cos x \) is also periodic with period \( 360^\circ \).
- \( \cos x = \cos (x + 360^\circ) = \cos (x + 720^\circ) = \cos (x + 360n^\circ) \), where \( n \) is an integer.
- It also bounces between \( y = -1 \) and \( y = 1 \), and it can never have a value outside this range.
- It crosses the \( y \)-axis at \( x = 0^\circ \) and the \( x \)-axis at \( 90^\circ, 270^\circ \) etc.
- \( \cos (-x) = \cos x \). The graph is symmetrical about the \( y \)-axis, so you could reflect it in the \( y \)-axis and it would look the same.
- The graph of \( y = \cos x \) looks like this:

\[
\begin{align*}
&m = 0 \\
&y = \cos x
\end{align*}
\]

\( \tan x \)

- The graph of \( y = \tan x \) is also periodic, but this time it repeats itself every \( 180^\circ \).
- So \( \tan x = \tan (x + 180^\circ) = \tan (x + 360^\circ) = \tan (x + 180n^\circ) \), where \( n \) is an integer.
- It takes values between \( -\infty \) and \( \infty \) in each \( 180^\circ \) interval.
- It goes through the origin (as \( x = 0^\circ = 0 \)).
- \( \tan \) is undefined at \( \pm 90^\circ, \pm 270^\circ, \pm 450^\circ \... \) — at these points it jumps from \( -\infty \) to \( \infty \) or vice versa. The lines \( x = \pm 90^\circ, \pm 270^\circ \) etc. are asymptotes — the graph never touches them, but gets infinitely close.
- The graph of \( y = \tan x \) looks like this:

\[
\begin{align*}
&m = 0 \\
&y = \tan x
\end{align*}
\]

You can apply the different types of transformations (see p.48-50) to the trig graphs as well.

\[\text{Example}\]

On the same axes, sketch the graphs of \( y = \cos x \) and \( y = -2 \cos x \) in the range \( -360^\circ \leq x \leq 360^\circ \).

1. Start by sketching the graph of \( \cos x \) (the grey line on the graph on the next page).
2. The transformation is in the form \( y = a \cos x \), so it will be stretched vertically.
3. \( a = -2 \), so it will be stretched by a factor of 2. As \( a \) is negative, it will also be reflected in the \( x \)-axis. Use this information to sketch the graph (see next page).
Exercise 3.1

Q1 On the same set of axes, sketch the graphs of \( y = \cos x \) and \( y = \cos x + 3 \) in the interval \(-360^\circ \leq x \leq 360^\circ\).

Q2 For the interval \(-180^\circ \leq x \leq 180^\circ\), sketch the graphs of:
   a) \( y = \cos x \) and \( y = \cos (x + 90^\circ) \)
   b) \( y = \sin x \) and \( y = \frac{1}{2} \sin x \)
   Sketch each pair of graphs on the same set of axes.

Q3 For the interval \(0^\circ \leq x \leq 360^\circ\), sketch the graphs of:
   a) \( y = \sin x \) and \( y = \sin 3x \)
   b) \( y = \cos x \) and \( y = -\cos x \)
   Sketch each pair of graphs on the same set of axes.

Q4 a) Sketch the graph of \( f(x) = \tan x \) in the interval \(-90^\circ \leq x \leq 270^\circ\).
   b) Translate this graph \(90^\circ\) to the left and sketch it on the same set of axes as part a).
   c) Write down the equation of the transformed graph.

Q5 a) Sketch the graph of \( y = \sin x \) in the interval \(-360^\circ \leq x \leq 360^\circ\).
   b) Sketch the graph horizontally by a factor of 2 and sketch it on the same set of axes as part a).
   c) Write down the equation of the transformed graph.

Q6 The diagram on the right shows the graph of \( y = \sin x \) and a transformed graph.
   a) Describe the transformation.
   b) Write down the equation of the transformed graph.

Q7 The diagram on the left shows the graph of \( y = \cos x \) and a transformed graph.
   a) Describe the transformation.
   b) Write down the equation of the transformed graph.

4. Solving Trig Equations

Sketching a graph

To solve trig equations in a given interval, draw a graph of the function and read solutions off the graph. You'll often find that there's more than one solution to the equation — in every \(360^\circ\) interval, there are usually two solutions to an equation, and if the interval is bigger, there'll be even more solutions.

Example

Solve \( \sin x = -0.3 \) for \(0^\circ \leq x \leq 720^\circ\). Give your answers to 3 s.f.

1. Use your calculator to work out the first value: \( \sin x = -0.3 \Rightarrow x = 174.5^\circ \ldots \).
   However, this is outside the given interval for \(x\), so add on \(360^\circ\) to find a solution in the interval:
   \[ -174.5^\circ + 360^\circ = 185.5^\circ \ldots \]

2. Sketch a graph of \( \sin x \) over the interval \(0^\circ \leq x \leq 720^\circ\), and draw a horizontal line across at \(y = -0.3\). There will be 2 repetitions of the sin wave in this interval.

3. There are 4 solutions in the given interval. To find the solution before 342.54\ldots, use the symmetry of the graph. This solution is 174.5\ldots away from 360\ldots. So the other solution between 0\ldots and 360\ldots will be 174.5\ldots away from 180\ldots:
   \[ 180^\circ + 174.5^\circ = 354.5^\circ \ldots \]

4. For the next two solutions (the ones between 360\ldots and 720\ldots), just add 360\ldots onto the values you've already found: 197.45\ldots + 360\ldots = 557.45\ldots and 342.54\ldots + 360\ldots = 702.54\ldots.

5. So the solutions to \( \sin x = -0.3 \) for \(0^\circ \leq x \leq 720^\circ\), to 3 s.f., are: \( x = 157^\circ, 347^\circ, 557^\circ, 703^\circ \)
Exercise 4.1

Q1 By sketching a graph, find all the solutions to the equations below in the interval $0^\circ \leq x \leq 360^\circ$.

Give your answers to 1 decimal place where appropriate.

- a) $\sin x = 0.75$
- b) $\cos x = 0.31$
- c) $\tan x = -1.5$
- d) $\sin x = -0.42$
- e) $\cos x = -0.56$
- f) $\tan x = -0.67$
- g) $\cos x = -\frac{1}{\sqrt{3}}$
- h) $\tan x = -\sqrt{2}$
- i) $\sin x = \frac{1}{2}$
- j) $\tan x = \frac{1}{\sqrt{3}}$
- k) $\cos x = \frac{1}{2}$
- l) $\cos x = \frac{\sqrt{3}}{2}$

Q2 One solution of $\cos x = -0.8$ is $143.1^\circ$ (1 d.p.).

Use the graph on the right to find all the solutions in the interval $0^\circ \leq x \leq 360^\circ$.

Q3 Find all the solutions of the equation $\tan x = 2.5$ in the interval $0^\circ \leq x \leq 1080^\circ$.

Give your answers to 1 decimal place.

Q4 Find all the solutions of the equation $\sin x = 0.81$ in the interval $-360^\circ \leq x \leq 360^\circ$.

Give your answers to 3 significant figures.

Using a CAST diagram

Another way to find the solutions to a trig equation is by using a CAST diagram. CAST stands for Cos, All, Sin, Tan, and it shows you where each of these functions is positive by splitting a $360^\circ$ period into quadrants.

For an acute angle $x$, make an angle of $x^\circ$ from the horizontal in each of the four quadrants. Ignore the ones that give a negative result — the other two values are your solutions between $0^\circ$ and $360^\circ$.

Tip: If the given result is negative, you need the two quadrants where the trig function is negative.

Example

Find all the solutions of $\tan x = -6$ for $0^\circ \leq x \leq 720^\circ$.

Give your answers to 1 d.p.

1. The first solution is $x = -80.5^\circ$ (1 d.p.). Ignore the negative and just put the value $80.5^\circ$ into the CAST diagram.
2. Add the same angle to each quadrant, measuring from the horizontal in each case.
3. $\tan x$ is negative in the second and fourth quadrants (i.e., the 'S' and 'C' quadrants).
4. There are two solutions: $99.5^\circ$ and $279.5^\circ$.

Exercise 4.2

Q1 One solution of $\sin x = 0.45$ is $x = 26.7^\circ$ (1 d.p.).

Use a CAST diagram to find all the solutions in the interval $0^\circ \leq x \leq 360^\circ$.

Q2 Use a CAST diagram to find the solutions of the following equations in the interval $0^\circ \leq x \leq 360^\circ$.

Give your answers to 1 d.p.

- a) $\cos x = 0$
- b) $\tan x = 2.7$
- c) $\sin x = -0.15$
- d) $\tan x = 0.3$
- e) $\tan x = -0.6$
- f) $\sin x = -0.29$
- g) $4 \sin x + 1 = 0$
- h) $4 \cos x - 3 = 0$
- i) $5 \tan x + 7 = 0$

Q3 Use a CAST diagram to find all the solutions to $\tan x = -8.4$ in the interval $0^\circ \leq x \leq 360^\circ$.

Give your answers to 3 s.f.

Q4 Use a CAST diagram to find all the solutions to $\sin x = 0.75$ in the interval $0^\circ \leq x \leq 720^\circ$.

Give your answers to 1 d.p.

Q5 Use a CAST diagram to find all the solutions to $\cos x = 0.31$ in the interval $-180^\circ \leq x \leq 180^\circ$.

Give your answers to 1 d.p.

Q6 Use a CAST diagram to find all the solutions to $\sin x = 0.82$ in the interval $0^\circ \leq x \leq 720^\circ$.

Give your answers to 3 s.f.

Q7 Use a CAST diagram to find all the solutions to $\cos x = -0.06$ in the interval $0^\circ \leq x \leq 1080^\circ$.

Give your answers to 1 d.p.

Q8 Use a CAST diagram to find all the solutions to $\tan x = 11.6$ in the interval $-360^\circ \leq x \leq 360^\circ$.

Give your answers to 1 d.p.
Changing the interval

Solving equations of the form $\sin kx = n$

- Multiply the interval you're looking for solutions in by $k$. E.g., for the equation $\sin 2x = n$ in the interval $0^\circ \leq x \leq 360^\circ$, you'd look for solutions in the interval $0^\circ \leq 2x \leq 720^\circ$.
- Then solve the equation over this new interval.
- This gives you solutions for $kx$, so you then need to divide each solution by $k$ to find the values of $x$.

You can either sketch the graph over the new interval (which will show you how many solutions there are) or you can use the CAST method to find solutions between $0^\circ$ and $360^\circ$ then add on multiples of $360^\circ$ until you have all the solutions in the new interval.

**Example**

Solve $\cos 4x = 0.6$ for $0^\circ \leq x \leq 360^\circ$. Give your answers to 1 d.p.

1. First, change the interval. $k = 4$, so multiply the whole interval by $4$: $0^\circ \leq 4x \leq 1440^\circ$
2. Then solve the equation to find the solutions for $4x$.
3. Find the first solution using a calculator:
   \[
   \cos 4x = 0.6 \Rightarrow 4x = 53.13^\circ \ (2 \text{ d.p.})
   \]
4. You want the quadrants where $\cos$ is positive, so the other solution between $0^\circ$ and $360^\circ$ is:
   \[
   4x = 360^\circ - 53.13^\circ = 306.87^\circ \ (2 \text{ d.p.)}
   \]
5. Now add on multiples of $360^\circ$ to find all the solutions in the interval $0^\circ \leq x \leq 1440^\circ$ (to 2 d.p.):
   \[
   53.13^\circ, 306.87^\circ, 413.13^\circ, 666.87^\circ, 773.13^\circ, 1026.87^\circ, 1133.13^\circ, 1386.87^\circ
   \]
6. These are solutions for $4x$. To find the solutions for $x$, divide through by 4.
   So the solutions to $\cos 4x = 0.6$ in the interval $0^\circ \leq x \leq 360^\circ$ are:
   \[
   13.3^\circ, 76.7^\circ, 103.3^\circ, 166.7^\circ, 193.3^\circ, 256.7^\circ, 283.3^\circ, 346.7^\circ
   \]

**Example**

Solve $\sin 3x = -\frac{1}{2}$ for $0^\circ \leq x \leq 360^\circ$.

1. $k = 3$, which means the interval you need to find solutions in is: $0^\circ \leq 3x \leq 1080^\circ$
   So sketch the graph of $y = \sin x$ between $0^\circ$ and $1080^\circ$ (see next page).
2. Find a solution: $3x = -60^\circ$. This is outside the interval for $3x$, so use the pattern of the graph to find a solution in the interval: $-60^\circ + 360^\circ = 300^\circ$
3. Use the symmetry of the graph to find the other solution between $0^\circ$ and $360^\circ$: $180^\circ + 45^\circ = 225^\circ$
4. The graph repeats every $360^\circ$, so add on all solutions to these values to find the other solutions between $0^\circ$ and $1080^\circ$: $3x = 585^\circ, 675^\circ, 945^\circ, 1035^\circ$

**Exercise 4.3**

In this exercise, give all answers to 1 d.p.

Q1 Solve the following equations in the interval $0^\circ \leq x \leq 360^\circ$:
   a) $\sin 2x = 0.6$
   b) $\tan 4x = 4.6$
   c) $\cos 3x = -0.24$
   d) $\sin 3x = 0.94$
   e) $\cos 5x = 0.5$
   f) $\tan 2x = -6.7$
Q2 Find all the solutions to $\cos 2x = 0.72$ in the interval $0^\circ \leq x \leq 360^\circ$.
Q3 Find all the solutions to $\frac{1}{2} \sin x = -0.61$ in the interval $0^\circ \leq x \leq 360^\circ$.
Q4 Solve $\tan 3x = 2.1$ in the interval $0^\circ \leq x \leq 360^\circ$.
Q5 Solve $\cos \frac{x}{2} = \frac{\sqrt{3}}{2}$ in the interval $-180^\circ \leq x \leq 180^\circ$.

Solving equations of the form $\sin (x + c) = n$

Here, instead of multiplying the interval, you have to add or subtract the value of $c$.

- Add (or subtract) the value of $c$ to the whole interval — so the interval $0^\circ \leq x \leq 360^\circ$ becomes $c \leq x + c \leq 360^\circ + c$.
- Now solve the equation over the new interval — either sketch a graph or use a CAST diagram.
- Finally, subtract (or add) $c$ from your solutions to give the values for $x$.

**Example**

Solve $\tan (x - 75^\circ) = 2$ for $0^\circ \leq x \leq 360^\circ$.

1. Find the new interval. Subtract $75^\circ$ from each bit of the interval: $-75^\circ \leq x - 75^\circ \leq 285^\circ$
   You'll need to sketch $\tan x$ over this interval.
2. Use your calculator to find a solution:
   $\tan (x - 75^\circ) = 2 \Rightarrow x - 75^\circ = 63.4^\circ$ (1 d.p.)
3. Use the pattern of the graph to find the other solution in the interval — add on 180° to the solution you’ve already found: 63.4° + 180° = 243.4° (1 d.p.)
4. Finally, add on 75° to find the solutions in the interval 0° ≤ x ≤ 360° (to 1 d.p.):
   \[ x = 138.4°, 318.4° \]

**Example**

Solve \( 2 \sin (x + 60°) + \sqrt{3} = 0 \) for \( 0° \leq x \leq 720° \).

1. First, rearrange: \( 2 \sin (x + 60°) + \sqrt{3} = 0 \) ⇒ \( 2 \sin (x + 60°) = -\sqrt{3} \) ⇒ \( \sin (x + 60°) = -\frac{\sqrt{3}}{2} \)
2. Then add 60° to each bit of the interval: \( 60° \leq x + 60° \leq 780° \)
3. \( \sin 60° = \frac{\sqrt{3}}{2} \), so put 60° into the CAST diagram:
   - \( \sin x \) is negative in the 3rd and 4th quadrants, so the solutions between 0° and 360° are:
     \[ x + 60° = 180° + 60° = 240° \text{ and } x + 60° = 360° - 60° = 300° \]
4. To find the other solutions, just add 360° to these solutions:
   \[ x + 60° = 600°, 660° \]
5. Finally, subtract 60° from each solution: \( x = 180°, 240°, 540°, 600° \)

**Exercise 4.4**

In this exercise, give all answers to 1 d.p.

Q1 Find all the solutions to \( \sin (x - 27°) = 0.64 \) in the interval \( 0° \leq x \leq 360° \).
Q2 Solve \( \tan (x - 140°) = -0.76 \) in the interval \( 0° \leq x \leq 360° \).
Q3 Find all the solutions to \( \tan (x + 36°) = 0.45 \) in the interval \( 0° \leq x \leq 360° \).
Q4 Find all the solutions to \( \tan (x + 73°) = 1.84 \) in the interval \( 0° \leq x \leq 360° \).
Q5 Find all the solutions to \( \sin (x - 45°) = -0.25 \) in the interval \( -180° \leq x \leq 360° \).
Q6 Solve \( \tan (x + 22.5°) = 0.13 \) in the interval \( 0° \leq x \leq 360° \).

**Using trig identities to solve equations**

Remember the trig identities from p.69: \( \tan x = \frac{\sin x}{\cos x} \) and \( \sin^2 x + \cos^2 x = 1 \)
You can use them to help solve complicated-looking trig equations — e.g. ones that contain two different trig functions. If you’re left with a quadratic equation (e.g. one that contains both \( \sin x \) and \( \sin x \)), you might need to factorise before you solve it — see p.16.

**Example**

Solve \( 6 \cos^2 x + \cos x \tan x = 5 \) for \( 0° \leq x \leq 360° \).

Give any non-exact answers to 1 d.p.

1. Use the identity for tan to replace the tan term:
   \[ 6 \cos^2 x + \cos x \tan x = 5 \Rightarrow 6 \cos^2 x + \cos x \frac{\sin x}{\cos x} = 5 \]
2. Cancel the cos term:
   \[ 6 \cos^2 x + \sin x = 5 \]
3. Replace the \( \cos x \) with \( 1 - \sin^2 x \):
   \[ 6(1 - \sin^2 x) + \sin x = 5 \]
4. Rearrange to form a quadratic equation in terms of \( \sin x \):
   \[ 6 \sin^2 x - \sin x - 1 = 0 \]
5. Factorise the quadratic:
   \[ (2 \sin x - 1)(3 \sin x + 1) = 0 \]
   Tip: Use the substitution \( y = \sin x \) to help factorise if you need to.
6. Solve the quadratic equation to get solutions for \( \sin x \):
   \[ 2 \sin x - 1 = 0 \Rightarrow \sin x = \frac{1}{2} \text{ or } 3 \sin x + 1 = 0 \Rightarrow \sin x = -\frac{1}{3} \]
7. One solution for \( \sin x = \frac{1}{2} \) is: \( x = 30° \)
   Use a CAST diagram to find the other solution — you want a positive solution, so look at the second quadrant: \( 180° - 30° = 150° \)
8. One solution for \( \sin x = -\frac{1}{3} \) is: \( x = -19.47...° \)
   Use a CAST diagram to find the solutions in the given interval — you need a negative solution for \( \sin x \), so look at the 3rd and 4th quadrants:
   \[ 180° + 19.47...° = 199.47...° \text{ and } 360° - 19.47...° = 340.52...° \]
   You now have all the solutions:
   \( x = 30°, 150°, 199.5° \text{ and } 340.5° \text{ (to 1 d.p.)} \)

**Exercise 4.5**

In this exercise, give any non-exact answers to 1 d.p.

Q1 Solve each of the following equations for values of \( x \) in the interval \( 0° \leq x \leq 360° \):
   a) \( \tan (x - 5°) \sin x - 1 = 0 \)
   b) \( 5 \sin x \tan x - 4 \tan x = 0 \)
   c) \( \tan^2 x = 9 \)
   d) \( 4 \cos^2 x = 3 \cos x \)
   e) \( 3 \sin x = 5 \cos x \)
   f) \( 5 \tan^2 x - 2 \tan x = 0 \)
   g) \( 6 \cos^2 x - \cos x - 2 = 0 \)
   h) \( 7 \sin x + 3 \cos x = 0 \)
Q2 Find the solutions to each of the following equations in the given interval:
   a) \( \tan x = \sin x \cos x \) \( 0° \leq x \leq 360° \)
   b) \( 5 \cos^2 x - 9 \sin x = 3 \) \( -360° \leq x \leq 720° \)
   c) \( 2 \sin^2 x + \sin x - 1 = 0 \) \( -360° \leq x \leq 360° \)
Q3 a) Show that the equation \( 4 \sin^2 x - 3 \cos x \) can be written as \( 4 \cos^2 x - 3 \cos x = -1 \).
   b) Hence solve the equation \( 4 \sin^2 x - 3 \cos x \) in the interval \( 0° \leq x \leq 360° \).
Q4 Find all the solutions of the equation \( 9 \sin^2 2x + 3 \cos 2x = 7 \) in the interval \( 0° \leq x \leq 360° \).
Q5 Find all the solutions of the equation \( \cos \frac{\cos x}{\cos x} + \sin x = 3 \) in the interval \( -360° \leq x \leq 360° \).
Q1 Write down the exact values of \( \cos 30^\circ, \sin 30^\circ, \tan 30^\circ, \cos 45^\circ, \sin 45^\circ, \tan 45^\circ, \cos 60^\circ, \sin 60^\circ \) and \( \tan 60^\circ \).

Q2 The points below lie on the unit circle. For each point, if a line from the origin to the point makes an angle of \( \theta \) when measured in an anticlockwise direction from the positive x-axis, find the exact value of \( \theta \), where \( 0 \leq \theta \leq 180^\circ \).
   a) \( \left( \frac{\sqrt{3}}{2}, \frac{1}{2} \right) \)  
   b) \( \left( \frac{\sqrt{3}}{2}, \frac{1}{2} \right) \)  
   c) \(-1, 0\)

Q3 For triangle \( \triangle ABC \), in which \( A = 30^\circ, C = 55^\circ \) and \( b = 6 \) m:
   a) Find all the sides and angles of the triangle.  
   b) Find the area of the triangle.

Q4 For triangle \( \triangle PQR \), in which \( p = 13 \) km, \( q = 23 \) km and \( R = 20^\circ \):
   a) Find all the sides and angles of the triangle.  
   b) Find the area of the triangle.

Q5 Find all the angles in the triangle below in degrees to 1 d.p.

Q6 Find the missing sides and angles for the 2 possible triangles \( \triangle ABC \) which satisfy \( b = 5, a = 3, \theta = 35^\circ \).

Q7 Show that \( \tan x - \sin x \cos x = \sin^2 x \tan x \).

Q8 Show that \( \tan^2 x - \cos^2 x + 1 = \tan^2 x (1 + \cos^2 x) \).

Q9 Simplify: \( \sin x + \cos x \tan x + \sin^2 x \).

Q10 Show that \( \frac{\sin x + \sin^2 x \cos x}{\cos x + 1} = -1 \).

Q11 Sketch the following graphs in the interval \(-360^\circ \leq x \leq 360^\circ\), making sure you label all of the key points.
   a) \( y = \cos x \)  
   b) \( y = \sin x \)  
   c) \( y = \tan x \)

Q12 Below is the graph of \( y = \cos x \) and a transformation of the graph.
   What is the equation of the transformed graph?

Q13 Below is a graph of \( y = \sin x \) and a transformation of the graph.
   What is the equation of the transformed graph?

Q14 Sketch the following pairs of graphs on the same axes:
   a) \( y = \cos x \) and \( y = \frac{1}{2} \cos x \) (for \( 0^\circ \leq x \leq 360^\circ \))
   b) \( y = \sin x \) and \( y = \sin (x + 30^\circ) \) (for \( 0^\circ \leq x \leq 360^\circ \))
   c) \( y = \tan x \) and \( y = \tan 3x \) (for \( 0^\circ \leq x \leq 180^\circ \))

Q15 a) Solve each of these equations for \( 0^\circ \leq \theta \leq 360^\circ \):
   (i) \( \sin \theta = \frac{\sqrt{3}}{2} \)  
   (ii) \( \tan \theta = -1 \)  
   (iii) \( \cos \theta = -\frac{1}{\sqrt{2}} \)
   b) Solve each of these equations for \(-180^\circ \leq \theta \leq 180^\circ \) (giving your answers to 1 d.p.):
   (i) \( \cos \theta = -\frac{1}{2} \)  
   (ii) \( \sin (\theta + 35^\circ) = 0.3 \)  
   (iii) \( \tan \theta = -500 \)

Q16 Find all the solutions to \( \sin^2 x = \cos x + 5 \) in the interval \( 0^\circ \leq x \leq 360^\circ \), giving your answers to 1 d.p. where appropriate.

Q17 Solve \( 3 \tan x + 2 \cos x = 0 \) for \(-90^\circ \leq x \leq 90^\circ \).

Q18 Find all the solutions of the equation \( 8 \sin^2 x + 2 \sin x - 1 = 0 \) in the interval \( 0^\circ \leq x \leq 360^\circ \), giving your answers to 1 d.p. where appropriate.

Q19 Find all the solutions of the equation \( \tan x - 3 \sin x = 0 \) in the interval \( 0^\circ \leq x \leq 720^\circ \), giving your answers to 1 d.p.
Chapter 18 — Trigonometry 2

You came across some trigonometry in Chapter 6, but there's quite a bit more here. One key difference in Year 2 trigonometry is that you don't just work in degrees — you have to work with radians too.

1. Arcs and Sectors

Radians

- A radian (rad) is just another unit of measurement for an angle.
- 1 radian is the angle formed in a sector that has an arc length that is the same as the radius. In other words, if you have a sector with an angle of 1 radian, then the length of the arc will be exactly the same length as the radius r.
- This is how radians relate to degrees:
  - $360^\circ$ (a complete circle) = $2\pi$ radians
  - $180^\circ$ = $\pi$ radians
  - 1 radian = $57.3^\circ$

  To convert from radians to degrees, divide by $\pi$, then multiply by 180.
  To convert from degrees to radians, divide by 180, then multiply by $\pi$.

Here's a table of some of the useful common angles, in degrees and radians:

<table>
<thead>
<tr>
<th>Degrees</th>
<th>0</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>90</th>
<th>120</th>
<th>180</th>
<th>270</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radians</td>
<td>$0$</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\pi}{2}$</td>
<td>$\pi$</td>
<td>$\frac{3\pi}{2}$</td>
<td>$2\pi$</td>
<td>$4\pi$</td>
</tr>
</tbody>
</table>

**Example**

Convert $\frac{\pi}{15}$ into degrees.

- To convert from radians to degrees, divide by $\pi$...
  $\frac{\pi}{15} \times \pi = \frac{1}{15}$
- ... then multiply by 180...
  $\frac{1}{15} \times 180 = 12^\circ$

**Exercise 1.1**

Q1 Convert the angles below into radians. Give your answers in terms of $\pi$.
   a) $180^\circ$  
   b) $210^\circ$  
   c) $270^\circ$  
   d) $70^\circ$  
   e) $150^\circ$  
   f) $75^\circ$

Q2 Convert the angles below into degrees.
   a) $\frac{\pi}{4}$  
   b) $\frac{\pi}{3}$  
   c) $\frac{\pi}{2}$  
   d) $\frac{5\pi}{6}$  
   e) $\frac{3\pi}{4}$  
   f) $\frac{7\pi}{3}$

Arc length and sector area

A sector is a part of a circle formed by two radii and part of the circumference. The arc of a sector is the curved edge of the sector. You can work out the length of the arc or the area of the sector if you know the angle at the centre ($\theta$) and the length of the radius ($r$).

Tips: When working out arc length and sector area you always work in radians.
For a circle with radius \( r \) and a sector with angle \( \theta \) (measured in radians), its arc length, \( s \), is given by:

\[ s = r\theta \]

For the same sector, its area, \( A \), is given by:

\[ A = \frac{1}{2} r^2 \theta \]

Tip: If you put \( \theta = 2\pi \) in either formula (and so make the sector equal to the whole circle), you get the normal circumference and area formulas.

**Examples**

a) Find the exact length \( L \) and area \( A \) in the diagram to the right.

1. Convert the angle to radians:
   \[ 45^\circ = \frac{45 \times \pi}{180} = \frac{\pi}{4} \text{ radians} \]
2. Then put everything in the formulas:
   \[ L = r\theta = 20 \times \frac{\pi}{4} = 5\pi \text{ cm} \]
   \[ A = \frac{1}{2} r^2 \theta = \frac{1}{2} \times 20^2 \times \frac{\pi}{4} = 50\pi \text{ cm}^2 \]

b) Find the exact value of \( \theta \) in the diagram to the left.

Use the formula for the arc length:

\[ s = r\theta \Rightarrow 4\pi = 20\theta \]

\[ \theta = \frac{4\pi}{20} = \frac{\pi}{5} \text{ radians} \]

**Exercise 1.2**

Q1 The diagram to the right shows a sector OAB. The centre is at O and the radius is 6 cm. The angle AOB is 2 radians. Find the arc length and area of this sector.

Q2 The diagram to the left shows a sector OPO. The centre is at O and the radius is 8 cm. The angle POQ is 46°. Find the arc length and area of this sector to 1 d.p.

Q3 A sector of a circle of radius 4 cm has an area of 6\( \pi \) cm\(^2\). Find the exact value of the angle \( \theta \).

Q4 The diagram below shows a sector of a circle with a centre O and radius \( r \) cm. The angle AOB shown is \( \theta \).

For each of the following values of \( \theta \) and \( r \), give the arc length and the area of the sector. Where appropriate give your answers to 3 s.f.

a) \( \theta = 1.2 \) radians, \( r = 5 \) cm
b) \( \theta = 0.6 \) radians, \( r = 4 \) cm
c) \( \theta = 80^\circ \), \( r = 9 \) cm
d) \( \theta = \frac{5\pi}{6} \), \( r = 4 \) cm

Q5 The diagram to the right shows a sector ABC of a circle, where the angle BAC is 0.9 radians. Given that the area of the sector is 16.2 cm\(^2\), find the arc length \( s \).

Q6 A circle C has a radius of length 3 cm with centre O. A sector of this circle is given by angle AOB which is 20°. Find the length of the arc AB and the area of the sector. Give your answer in terms of \( \pi \).

Q7 The sector shown on the left has an arc length of 7 cm. The angle AML is 1.4 rad. Find the area of the sector.

Q8 A circle of radius \( r \) contains a sector of area 80\( \pi \) cm\(^2\). Given that the arc length of the sector is 16 cm, find the angle of the sector \( \theta \) and the value of \( r \), giving your answers to 3 s.f.

Q9 The diagram to the right shows a semicircle of radius 2 cm, with a smaller sector of radius 1 cm removed. Given that the area of the sector A and the area of B are equal, find the exact value of \( \theta \).

**2: Small Angle Approximations**

When \( \theta \) (measured in radians) is very small, you can approximate the value of \( \sin \theta \), \( \cos \theta \) and \( \tan \theta \) using the small angle approximations:

\[ \sin \theta \approx \theta \quad \cos \theta \approx 1 - \frac{1}{2} \theta^2 \quad \tan \theta \approx \theta \]

Tip: These approximations only work when \( \theta < 1 \).

**Example** Give an approximation for \( \cos 0.2 \).

Use the small angle approximation for \( \cos \) and \( \sin \) to get \( \cos \theta \approx 1 - \frac{1}{2} \theta^2 \).

Tip: The actual value of \( \cos 0.2 \) is 0.980067 (to 6 d.p.), so the approximation is pretty accurate.

These approximations can be used to approximate more complicated functions, which could involve \( \sin \), \( \cos \) and \( \tan \) of multiples of \( \theta \). Make sure that you apply the approximation to everything inside the trig function — for example:

\[ \tan 4\theta = 4\theta \quad \sin \frac{1}{2} \theta = \frac{1}{2} \theta \quad \cos 3\theta = 1 - \frac{3}{2} \theta^2 \]